# Biholomorphic inequivalence of the euclidean ball and the poly-disc

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Again, what we did not finish yesterday

the final act - curvature inequality for contractions

the Bergman kernel

transformation rule for the kernel

the euclidean ball and the polydisc are not biholomorphic

new quasi-invariant kernels from old ones

# Again, what we did not finish yesterday

# polarization

If $\gamma$ is holomorphic and admits the power series expansion $\gamma(w) = \zeta_0 + \zeta_1 w + \zeta_2 w^2 + \cdots,$ then the norm $\|\gamma(w)\|^2$ is a function of $w$ and $\bar{w}$. It has the form

$$\sum_{j,k=0}^{\infty} \langle \zeta_j, \zeta_k \rangle w^j \bar{w}^k, \ \zeta_0, \zeta_2, \ldots \in \mathscr{H}.$$

Polarizing $\|\gamma(w)\|^2$, we obtain a new function $\tilde{\gamma}(z,w) := \langle \gamma(z), \gamma(w) \rangle$.

Thus $((\tilde{\gamma}(z_i, z_j)))$ is non negative definite for all choices of $z_1, \ldots z_n$ in $\mathbb{D}$. This is just the positive-definiteness of the kernel function $K(z,w) = \langle \gamma(z), \gamma(w) \rangle$!

The curvature $\mathscr{K}$ is a real analytic function and we have shown that $-\mathscr{K}$ is positive.

Let $\widetilde{\mathscr{K}}(z,w) := -\frac{\partial^2}{\partial \bar{w} \partial z} \log \tilde{\gamma}(z,w)$ denote the function obtained from polarization of the curvature $\mathscr{K}$.

What about positive definiteness of $-\widetilde{\mathscr{K}}$?

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Refining the computation that established the positivity of $\mathscr{K}$, we obtain a stronger inequality. Set

$$\varphi(w) := K(\cdot,w)\otimes\bar\partial K(\cdot,w) - \bar\partial K(\cdot,w)\otimes K(\cdot,w).$$

Note that $\varphi(w)\in\mathscr{H}$, $w\in\mathbb{D}$.

Moreover, a straightforward computation using the reproducing property of $K$ shows that

$$\begin{aligned}\langle\varphi(z),\varphi(w)\rangle &= \|\frac{\partial}{\partial w}\gamma(w)\|^2\|\gamma(w)\|^2 - |\langle\frac{\partial}{\partial w}\gamma(w),\gamma(w)\rangle|^2\\ &= \|\varphi(w)\|^4\frac{\partial^2}{\partial\bar w\partial w}\log\|\varphi(w)\|^{-2}\\ &= -\mathscr{G}_{K^{-2}}(w),\end{aligned}$$

where $\gamma(w) = K(\cdot,w)$, as before and $\mathscr{G}_{K^{-2}}$ is the Gaussian curvature of the metric $K(w,w)^{-2}$.

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# the final act - curvature inequality for contractions

## inequality for the Gaussian curvature

### Proposition

Let $T \in B_1(\mathbb{D})$ be a contraction. Assume that $T$ is unitarily equivalent to the operator $M^*$ on $(\mathscr{H}, K)$ for some non-negative definite kernel $K$ on the unit disc. Then the following inequality holds:

$$K^2(z,w) \preceq \mathbb{S}_{\mathbb{D}}^{-2}(z,w)\mathscr{G}_{K^{-1}}(z,w),$$

that is, the matrix

$$\Big(\Big(\mathbb{S}_{\mathbb{D}}^{-2}(w_i,w_j)\mathscr{G}_{K^{-1}}(w_i,w_j) - K^2(w_i,w_j)\Big)\Big)_{i,j=1}^{n}$$

is non-negative definite for every subset $\{w_1, \ldots, w_n\}$ of $\mathbb{D}$ and $n \in \mathbb{N}$.

# the Bergman kernel

For the Bergman space $\mathbb{A}^2(\mathbb{D}^m)$, of the polydisc $\mathbb{D}^m$, the orthonormal basis is $\{\sqrt{\prod_{k=1}^m (i_k+1)}\, z^I : I = (i_1, \ldots, i_m)\}$. Clearly, we have

$$B_{\mathbb{D}^m}(z,w) = \sum_{|I|=0}^{\infty} \Big(\prod_{k=1}^{m} (i_k+1)\Big) z^I \bar{w}^I = \prod_{i=1}^{m} (1 - z_i \bar{w}_i)^{-2}.$$

Similarly, for the Bergman space of the ball $\mathbb{A}^2(\mathbb{B}^m)$, the orthonormal basis is $\{\sqrt{\binom{-m-1}{|I|}\binom{|I|}{I}}\, z^I : I = (i_1, \ldots, i_m)\}$. Again, it follows that

$$B_{\mathbb{B}^m}(z,w) = \sum_{|I|=0}^{\infty} \binom{-m-1}{\ell} \Big( \sum_{|I|=\ell} \binom{|I|}{I} z^I \bar{w}^I \Big) = (1 - \langle z, w \rangle)^{-m-1}.$$

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# quasi-invariance of $B$

Any bi-holomorphic map $\varphi : \mathscr{D} \to \tilde{\mathscr{D}}$ induces a unitary operator $U_\varphi : \mathbb{A}^2(\tilde{\mathscr{D}}) \to \mathbb{A}^2(\mathscr{D})$ defined by the formula

$$(U_\varphi f)(z) = J(\varphi, z)\,\big(f \circ \varphi\big)(z),\; f \in \mathbb{A}^2(\tilde{\mathscr{D}}),\; z \in \mathscr{D}.$$

This is an immediate consequence of the change of variable formula for the volume measure on $\mathbb{C}^n$ :

$$\int_{\tilde{\mathscr{D}}} f\, dV = \int_{\mathscr{D}} (f \circ \varphi)\, |J_{\mathbb{C}}\varphi|^2 dV.$$

Consequently, if $\{\tilde{e}_n\}_{n\geq 0}$ is any orthonormal basis for $\mathbb{A}^2(\tilde{\mathscr{D}})$, then $\{e_n\}_{n\geq 0}$, where $\tilde{e}_n = J(\varphi, \cdot)(\tilde{e}_n \circ \varphi)$ is an orthonormal basis for the Bergman space $\mathbb{A}^2(\tilde{\mathscr{D}})$.

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# transformation rule for the kernel

Expressing the Bergman kernel $B_{\mathscr{D}}$ of the domains $\mathscr{D}$ as the infinite sum $\sum_{n=0}^{\infty} e_n(z)\overline{e_n(w)}$ using the orthonormal basis in $\mathbb{A}^2(\mathscr{D})$, we see that the Bergman Kernel $B$ is quasi-invariant, that is, If $\varphi : \mathscr{D} \to \widetilde{\mathscr{D}}$ is holomorphic then we have the transformation rule

$$J(\varphi, z) B_{\widetilde{\mathscr{D}}}(\varphi(z), \varphi(w)) \overline{J(\varphi, w)} = B_{\mathscr{D}}(z, w),$$

where $J(\varphi, w)$ is the Jacobian determinant of the map $\varphi$ at $w$.

If $\mathscr{D}$ admits a transitive group of bi-holomorphic automorphisms, then this transformation rule gives an effective way of computing the Bergman kernel. Thus

$$B_{\mathscr{D}}(z, z) = |J(\varphi_z, z)|^2 B_{\mathscr{D}}(0, 0),\ z \in \mathscr{D},$$

where $\varphi_z$ is the automorphism of $\mathscr{D}$ with the property $\varphi_z(z) = 0$.

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The quasi-invariance of the Bergman kernel $B_{\mathscr{D}}(z;w)$ also leads to a bi-holomorphic invariant for the domain $\mathscr{D}$. Setting

$$\mathscr{K}_{B_{\mathscr{D}}}(z) = \big(\tfrac{\partial^2}{\partial z_i \partial \bar{z}_j} \log B_{\mathscr{D}}\big)(z)$$

to be the curvature of the metric $B_{\mathscr{D}}(z,z)$, the function

$$\mathbb{I}_{\mathscr{D}}(z) := \frac{\det \mathscr{K}_{B_{\mathscr{D}}}(z)}{B_{\mathscr{D}}(z,z)},\ z \in \mathscr{D}$$

is a bi-holomorphic invariant for the domain $\mathscr{D}$.

Consider the special case, where $\varphi : \mathscr{D} \to \mathscr{D}$ is an automorphism. Clearly, in this case, $(U_{\varphi^{-1}} f)(z) := J(\varphi, z)(f \circ \varphi)(z)$ is unitary on $\mathbb{A}^2(\mathscr{D})$ for all $\varphi \in \mathrm{Aut}(\mathscr{D})$.

The map $J : \mathrm{Aut}(\mathscr{D}) \times \mathscr{D} \to \mathbb{C}$ satisfies the cocycle property, namely

$$J(\psi\varphi, z) = J(\varphi, \psi(z))J(\psi, z),\ \varphi, \psi \in \mathrm{Aut}(\mathscr{D}),\ z \in \mathscr{D}.$$

This makes the map $\varphi \to U_\varphi$ a homomorphism.

Thus we have a unitary representation of the Lie group $\mathrm{Aut}(\mathscr{D})$ on $\mathbb{A}^2(\mathscr{D})$.

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# the proof that $\mathbb{I}_{\mathscr{D}}$ is an invariant

Let $\varphi : \mathscr{D} \to \tilde{\mathscr{D}}$ be a bi-holomorphic map. Applying the change of variable formula twice to the function $\log B_{\tilde{\mathscr{D}}}(\varphi(z), \varphi(w))$, we have

$$\Big(\Big(\frac{\partial^2}{\partial z_i \partial \bar{w}_j} \log B_{\tilde{\mathscr{D}}}(\varphi(z), \varphi(w))\Big)\Big)_{ij} =$$

$$\Big(\Big(\frac{\partial \varphi_\ell}{\partial z_i}\Big)\Big)_{i\ell} \Big(\Big(\Big(\frac{\partial^2}{\partial z_\ell \partial \bar{w}_k} \log B_{\mathscr{D}}\Big)(z,w)\Big)\Big)_{\ell k} \Big(\Big(\frac{\partial \bar{\varphi}_k}{\partial \bar{z}_j}\Big)\Big)_{kj}.$$

Hence we conclude that $\mathscr{K}_{B_{\mathscr{D}}}$ is quasi-invariant under a bi-holomorphic map $\varphi$, namely,

$$J\varphi(w)^\sharp \mathscr{K}_{\tilde{\mathscr{D}}}(\varphi(w), \varphi(w)) \overline{J\varphi(w)} = \mathscr{K}_{\mathscr{D}}(w,w),\ w \in \mathscr{D}.$$

Also, the Bergman kernel $B_{\mathscr{D}}$ transforms according to the rule

$$\det J\varphi(w) B_{\tilde{\mathscr{D}}}(\varphi(w), \varphi(w)) \overline{\det J\varphi(w)} = B_{\mathscr{D}}(w,w).$$

Combining these we obtain we obtain a biholomorphic invariant for the domain $\mathscr{D}$.

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Taking determinants on both sidesof the transformation rule for the curvature, we get

$$\det \mathscr{K}_{\mathscr{D}}(w,w) = |J\varphi(z)|^2 \det \mathscr{K}_{\tilde{\mathscr{D}}}(\varphi(w),\varphi(w)).$$

Thus we get the invariance of $\mathbb{I}_{\mathscr{D}}$ :

$$\begin{aligned}
\frac{\det \mathscr{K}_{\mathscr{D}}(w,w)}{B_{\mathscr{D}}(w,w)} &= \frac{|J\varphi(z)|^2 \det \mathscr{K}_{\tilde{\mathscr{D}}}(\varphi(w),\varphi(w))}{B_{\mathscr{D}}(w,w)} \\
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Theorem

For any homogeneous domain $\mathscr{D}$ in $\mathbb{C}^n$, the function $\mathbb{I}_{\mathscr{D}}(z)$ is constant.

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the euclidean ball and the polydisc are not biholomorphic

# proof of the theorem

Since $\mathscr{D} \subseteq \mathbb{C}^n$ is homogeneous, it follows that there exists a bi-holomorphic map $\varphi_u$ of $\mathscr{D}$ for each $u \in \mathscr{D}$ such that $\varphi_u(0) = u$. Applying the transformation rule for $\mathbb{I}$, we have

$$\begin{aligned}
\mathbb{I}_{\mathscr{D}}(0) &= \frac{\det \mathscr{K}_{\mathscr{D}}(0,0)}{B_{\mathscr{D}}(0,0)} \\
&= \frac{\det \mathscr{K}_{\mathscr{D}}(\varphi_u(0), \varphi_u(0))}{B_{\mathscr{D}}(\varphi_u(0), \varphi_u(0))} \\
&= \frac{\det \mathscr{K}_{\mathscr{D}}(u,u)}{B_{\mathscr{D}}(u,u)} = \mathbb{I}_{\mathscr{D}}(u),\ u \in \mathscr{D}
\end{aligned}$$

It is easy to compute $\mathbb{I}_{\mathscr{D}}(0)$ when $\mathscr{D}$ is the bi-disc and the Euclidean ball in $\mathbb{C}^2$. For these two domains, it has the value $4$ and $9$ respectively. We conclude that these domains therefore can't be bi-holomorphically equivalent!

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$$\begin{aligned}\mathbb{I}_{\mathscr{D}}(0) &= \frac{\det \mathscr{K}_{\mathscr{D}}(0,0)}{B_{\mathscr{D}}(0,0)} \\ &= \frac{\det \mathscr{K}_{\mathscr{D}}(\varphi_u(0),\varphi_u(0))}{B_{\mathscr{D}}(\varphi_u(0),\varphi_u(0))} \\ &= \frac{\det \mathscr{K}_{\mathscr{D}}(u,u)}{B_{\mathscr{D}}(u,u)} = \mathbb{I}_{\mathscr{D}}(u),\ u \in \mathscr{D}\end{aligned}$$

It is easy to compute $\mathbb{I}_{\mathscr{D}}(0)$ when $\mathscr{D}$ is the bi-disc and the Euclidean ball in $\mathbb{C}^2$. For these two domains, it has the value $4$ and $9$ respectively. We conclude that these domains therefore can't be bi-holomorphically equivalent!

# new quasi-invariant kernels from old ones

Let $K$ be a complex valued positive definite kernel on $\mathscr{D}$. For $w$ in $\mathscr{D}$, and $p$ in the set $\{1,\ldots,d\}$, let $e_p : \Omega \to \mathscr{H}$ be the antiholomorphic function:

$$e_p(w) := K_w(\cdot) \otimes \frac{\partial}{\partial \bar{w}_p} K_w(\cdot) - \frac{\partial}{\partial \bar{w}_p} K_w(\cdot) \otimes K_w(\cdot).$$

Setting $G(z,w)_{p,q} = \langle e_p(w), e_q(z) \rangle$, we have

$$\frac{1}{2} G(z,w)_{p,q}{}^{\sharp} = K(z,w) \frac{\partial^2}{\partial z_q \partial \bar{w}_p} K(z,w) - \frac{\partial}{\partial \bar{w}_p} K(z,w) \frac{\partial}{\partial z_q} K(z,w)).$$

The curvature $\mathsf{K}$ of the metric $K$ is given by the $(1,1)$ - form $\sum \frac{\partial^2}{\partial w_q \partial \bar{w}_p} \log K(w,w) dw_q \wedge d\bar{w}_p$. Set

$$\mathscr{K}_K(z,w) := \left(\left( \frac{\partial^2}{\partial z_q \partial \bar{w}_p} \log K(z,w) \right)\right)_{qp}.$$

We note that $K(z,w)^2 \mathscr{K}(z,w) = \frac{1}{2} G(z,w)^{\sharp}$. Hence $K(z,w)^2 \mathscr{K}(z,w)$ defines a positive definite kernel on $\mathscr{D}$ taking values in $\mathrm{Hom}(V,V)$.

## rewrite the transformation rule

Or equivalently,

$$\begin{aligned}\mathscr{K}(\varphi(z),\varphi(w)) &= D\varphi(z)^{\sharp^{-1}}\mathscr{K}(z,w)\overline{D\varphi(z)}^{-1}\\ &= D\varphi(z)^{\sharp^{-1}}\mathscr{K}(z,w)\left(D\varphi(w)^{\sharp^{-1}}\right)^{*}\\ &= m_0(\varphi,z)\mathscr{K}(z,w)m_0(\varphi,w)^{*},\end{aligned}$$

where $m_0(\varphi,z) = D\varphi(z)^{\sharp^{-1}}$ and multiplying both sides by $K^2$, we have

$$K(\varphi(z),\varphi(w))^2\mathscr{K}(\varphi(z),\varphi(w)) = m_2(\varphi,z)K(z,w)^2\mathscr{K}(z,w)m_2(\varphi,w)^{*},$$

where $m_2(\varphi,z) = \left(\det_{\mathbb{C}} D\varphi(w)^2 D\varphi(z)^{\sharp}\right)^{-1}$ is a multiplier. Of course, we now have that

(i) $K^{2+\lambda}(z,w)\mathscr{K}(z,w)$, $\lambda > 0$, is a positive definite kernel and

(ii) it transforms with the co-cycle

$m_\lambda(\varphi,z) = \left(\det_{\mathbb{C}} D\varphi(z)^{2+\lambda} D\varphi(z)^{\dagger}\right)^{-1}$ in place of $m_2(\varphi,z)$.

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Thank You!